SOME PROBLEMS OF THE THEORY OF DYNAMIC PROGRAMMING FOR NONLINEAR SAMPLED DATE SYSTEMS

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This paper considers the problem of the choice of the control forces by means of which one can ensure either the realization of the law of motion in the phase space (or subspace) of the nonlinear sampled data system or the passage of the nonlinear sampled data system through predetermined states at given moments of time.

This paper concerns aspects of the theory of dynamic programming [1] related to the realization of the chosen strategy of controlling the motion.

For nonlinear continuous systems, the similar problem was considered by the author [2] in previous work.

1. The motion of a nonlinear sampled data system can be described by the following system of nonlinear difference equations:

$$\sum_{k=1}^{n} f_{jk}(T)\, y_k = x_j(t) + q_j(t) + \tag{1.1}$$

$$+ \psi_j(y_1, Ty_1, \ldots, T^{m_1-1}y_1, \ldots, y_n, Ty_n, \ldots, T^{m_n-1}y_n, t) \qquad (j = 1, \ldots, n)$$

Here y_k are the generalized coordinates of the system, $x_j(t)$ are the given external forces, $q_j(t)$ are the additional external forces for which the law of variation with respect to time must be chosen, such that the prescribed motion will take place. By $f_{jk}(T)$ are denoted polynomials in T, the coefficients of which are given functions of time, where T represents the lead operator, defined by the relation

$$T^{\mu} y_k = y_k(t + \mu\tau) \tag{1.2}$$

in which τ is some constant quantity. The highest degree T of the polynomials $f_{jk}(T)$ $(j = 1, \ldots, n)$ for a given k is represented by m_k.

The functions ψ_j $(j = 1, \ldots, n)$ on the right-hand side of Equation (1.1) appear as some nonlinear functions of their arguments. We shall assume that these functions are continuous with respect to all their arguments in a closed domain, and satisfy in that domain Lipschitz conditions relative to the arguments

$$y_1, Ty_1, \ldots, T^{m_1-1}y_1, \ldots, y_n, Ty_n, \ldots, T^{m_n-1}y_n$$

Let us note that the controlled systems, for which the presence of forces appearing as functions of the error is specified, are also described by Equation (1.1). The specified forces are considered as left-hand sides of Equations (1.1), the nonlinear functions ψ_j and the given external forces $x_j(t)$.

Equations (1.1) can be expressed [3] in the form

$$Tz_\nu + \sum_{k=1}^{r} a_{\nu k}(t) z_k = X_\nu(t) + Q_\nu(t) + \Psi_\nu(z_1, \ldots, z_r, t) \qquad (\nu = 1, \ldots, r) \quad (1.3)$$

Here

$$z_1 = y_1,\ z_2 = Ty_1, \ldots, z_{m_1} = T^{m_1-1}y_1, \ldots, z_r = T^{m_n-1}y_n \quad (1.4)$$

$$r = m_1 + m_2 + \ldots + m_n \quad (1.5)$$

$$X_{\sigma_j}(t) = \sum_{k=1}^{n} \frac{B_{kj}(t)}{\Delta^*(t)} x_k(t), \qquad Q_{\sigma_j}(t) = \sum_{k=1}^{n} \frac{B_{kj}(t)}{\Delta^*(t)} q_k(t)$$

$$\Psi_{\sigma_j}(z_1, \ldots, z_r, t) = \sum_{k=1}^{n} \frac{B_{kj}(t)}{\Delta^*(t)} \psi_k(z_1, \ldots, z_r, t) \qquad (\sigma_j = \sigma_1, \ldots, \sigma_n) \quad (1.6)$$

$$\sigma_1 = m_1,\ \ \sigma_2 = m_1 + m_2, \ldots, \sigma_n = r \quad (1.7)$$

and the functions $X_\mu(t)$, $Q_\mu(t)$, $\Psi_\mu(z_1, \ldots, z_r, t)$ for which $\mu \neq \sigma_l$ $(l = 1, \ldots, n)$ are identically equal to zero.

In the Expressions (1.6) we have designated by $\Delta^*(t)$ the determinant of the coefficients $b_{jk}(t)$, of the $T^{m_k}y_k(t)$ in the left-hand sides of Equations (1.1)

$$\Delta^*(t) = |b_{jk}(t)| \quad (1.8)$$

where it has been assumed that this determinant is not identically equal to zero. The minors of the elements b_k in the determinants (1.8) are denoted by B_{kj}.

The system of scalar differences (1.3) can be replaced by the matrix difference equation

$$z(t+\tau)+a(t)z(t)=X(t)+Q(t)+\Psi(z_1(t),\ldots,z_r(t),t) \qquad (1.9)$$

where

$$z(t)=\|z_\nu(t)\|,\quad a(t)=\|a_{\nu k}(t)\|,\quad X(t)=\|X_\nu(t)\|$$
$$Q(t)=\|Q_\nu(t)\|,\quad \Psi(z_1(t),\ldots,z_r(t),t)=\|\Psi_\nu(z_1(t),\ldots,z_r(t),t)\| \qquad (1.10)$$

Let us denote by $\theta(t)$ the fundamental matrix for the homogeneous matrix equation

$$z(t+\tau)+a(t)z(t)=0 \qquad (1.11)$$

The columns of the matrix $\theta(t)$ will be linearly independent of the particular solutions of Equation (1.11). Therefore the matrix $\theta(t)$ satisfies the relation

$$\theta(t+\tau)+a(t)\theta(t)=0 \qquad (1.12)$$

A solution is sought for the nonlinear matrix equation (1.9) of the form

$$z(t)=\theta(t)\chi(t) \qquad (1.13)$$

where $\chi(t)$ is a column matrix depending upon the given conditions. Substituting Expression (1.13) into Equation (1.9), we get

$$\theta(t+\tau)\chi(t+\tau)+a(t)\theta(t)\chi(t)=X(t)+Q(t)+\Psi(z_1(t),\ldots,z_r(t),t)$$

or

$$\theta(t+\tau)[\chi(t)+\chi(t+\tau)-\chi(t)]+a(t)\theta(t)\chi(t)=$$
$$=X(t)+Q(t)+\Psi(z_1(t),\ldots,z_r(t),t) \qquad (1.14)$$

Taking relations (1.12) into account, we get

$$\theta(t+\tau)\Delta\chi(t)=X(t)+Q(t)+\Psi(z_1(t),\ldots,z_r(t),t)$$

from where it follows, that

$$\Delta\chi(t) = \theta^{-1}(t+\tau)\,[X(t) + Q(t) + \Psi(z_1(t), \ldots, z_r(t), t)] \quad (1.15)$$

where $\Delta\chi(t)$ is the first difference of the function $\chi(t)$, and $\theta^{-1}(t)$ represents the inverse matrix of the matrix $\theta(t)$. From (1.15) it follows [1] that

$$\chi(t) = \sum_{i=1}^{\vartheta} \theta^{-1}(t+\tau-i\tau)\,[X(t-i\tau) + Q(t-i\tau) +$$
$$+ \Psi(z_1(t-i\tau), \ldots, z_r(t-i\tau), t-i\tau)] + A(t) \qquad \left(\vartheta = \left[\frac{t}{\tau}\right]\right) \quad (1.16)$$

where ϑ represents the fraction t/τ and $A(t)$ is a periodic function of period τ depending on the given conditions. Replacing the index of summation i by the relation $i = \vartheta - j + 1$, the Expression (1.16) is transformed to the form

$$\chi(t) = \sum_{j=1}^{\vartheta} \theta^{-1}(t-\vartheta\tau+j\tau)\,[X(t-\vartheta\tau+j\tau-\tau) + Q(t-\vartheta\tau+j\tau-\tau) +$$
$$+ \Psi(z_1(t-\vartheta\tau+j\tau-\tau), \ldots, z_r(t-\vartheta\tau+j\tau-\tau), t-\vartheta\tau+j\tau-\tau)] +$$
$$+ A(t) \quad (1.17)$$

Substituting Expression (1.17) into (1.13) gives

$$z(t) = \sum_{j=1}^{\vartheta} \theta(t)\,\theta^{-1}(t-\vartheta\tau+j\tau)[X(t-\vartheta\tau+j\tau-\tau) + Q(t-\vartheta\tau+j\tau-\tau) +$$
$$+ \Psi(z_1(t-\vartheta\tau+j\tau-\tau), \ldots, z_r(t-\vartheta\tau+j\tau-\tau), t-\vartheta\tau+j\tau-\tau)] +$$
$$+ \theta(t)\,A(t) \quad (1.18)$$

Let us now denote by $z^*(t) = \| z^*_\nu(t) \|$ the given matrix, which is determined in the time interval $0 < t < \tau$ by the law of variation of the unknown functions $z_\nu(t)$ $(\nu = 1, \ldots, r)$ in this (initial) interval.

In the interval of time $0 < t < \tau$, the first component on the right-hand side of relation (1.18) becomes zero. In order that the second component on the right-hand side of (1.18) coincide with $z^*(t)$ in that interval of time, it is necessary to choose as a periodic function $A(t)$ the following function

$$A(t) = \theta^{-1}(t-\vartheta\tau)\,z^*(t-\vartheta\tau) \quad (1.19)$$

With such a choice of the function $A(t)$, relation (1.18) becomes

$$z(t) = \theta(t)\,\theta^{-1}(t-\vartheta\tau)\,z^*(t-\vartheta\tau) + \sum_{j=1}^{\vartheta} \theta(t)\,\theta^{-1}(t-\vartheta\tau+j\tau)[X(t-\vartheta\tau+j\tau-\tau) + Q(t-\vartheta\tau+j\tau-\tau) + \Psi(z_1(t-\vartheta\tau+j\tau-\tau), \ldots, z_r(t-\vartheta\tau+j\tau-\tau), t-\vartheta\tau+j\tau-\tau)] \quad (1.20)$$

Introducing the function

$$N(t, j\tau) = \theta(t)\,\theta^{-1}(t-\vartheta\tau+j\tau) \quad (1.21)$$

which represents the weighting function for the matrix difference equation (1.11), relation (1.20) can be presented as

$$z(t) = N(t,0)\,z^*(t-\vartheta\tau) + \sum_{j=1}^{\vartheta} N(t,j\tau)\,X(t-\vartheta\tau+j\tau-\tau) + \sum_{j=1}^{\vartheta} N(t,j\tau)\,Q(t-\vartheta\tau+j\tau-\tau) + \sum_{j=1}^{\vartheta} N(t,j\tau) \times \Psi(z_1(t-\vartheta\tau+j\tau-\tau), \ldots, z_r(t-\vartheta\tau+j\tau-\tau), t-\vartheta\tau+j\tau-\tau) \quad (1.22)$$

The matrix relation (1.22) is equivalent to the matrix difference equation (1.9), together with the given law of variation of the unknown functions $z_\nu(t)$ ($\nu = 1, \ldots, r$) in the initial interval of time $0 < t < \tau$. Relation (1.22) is analogous to the matrix integral equation in the continuous analysis.

Since the functions $X_\mu(t)$, $Q_\mu(t)$, $\Psi_\mu(z_1(t), \ldots, z_r(t), t)$ for which $\mu \neq \sigma_l$ ($l = 1, \ldots, n$) are identically equal to zero, the system of scalar relations equivalent to the matrix equation (1.22) appears as

$$z_\nu(t) = \sum_{k=1}^{r} N_{\nu k}(t,0)\,z_k^*(t-\vartheta\tau) + \sum_{i=1}^{n}\sum_{j=1}^{\vartheta} N_{\nu\sigma_i}(t,j\tau) \times [X_{\sigma_i}(t-\vartheta\tau+j\tau-\tau) + Q_{\sigma_i}(t-\vartheta\tau+j\tau-\tau)] + \sum_{i=1}^{n}\sum_{j=1}^{\vartheta} N_{\nu\sigma_i}(t,j\tau) \times \Psi_{\sigma_i}(z_1(t-\vartheta\tau+j\tau-\tau), \ldots, z_r(t-\vartheta\tau+j\tau-\tau), t-\vartheta\tau+j\tau-\tau)$$

$$(\nu = 1, \ldots, r) \quad (1.23)$$

Let us now introduce the notations

$$W_{\nu l}(t;\ j\tau)=\sum_{i=1}^{n} N_{\nu\sigma_i}(t,\ j\tau)\,\frac{B_{li}(t-\vartheta\tau+j\tau-\tau)}{\Delta^*(t-\vartheta\tau+j\tau-\tau)} \tag{1.24}$$

$$(\nu=1,\ \ldots,\ r;\quad l=1,\ \ldots,\ n)$$

$$g_\nu(t)=\sum_{k=1}^{r} N_{\nu k}(t,\ 0)\, z_k^*(t-\vartheta\tau)+\sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{\nu l}(t,\ j\tau)\, x_l(t-\vartheta\tau+j\tau-\tau)$$

$$(\nu=1,\ \ldots,\ r) \tag{1.25}$$

Substituting Expressions (1.6) for X_{σ_i}, Q_{σ_i}, Ψ_{σ_i} into (1.23), it can be expressed as

$$z_\nu(t)=g_\nu(t)+\sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{\nu l}(t,\ j\tau)\, q_l(t-\vartheta\tau+j\tau-\tau)+\sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{\nu l}(t, j\tau)\times$$
$$\times\, \psi_l(z_1(t-\vartheta\tau+j\tau-\tau),\ \ldots,\ z_r(t-\vartheta\tau+j\tau-\tau),\ t-\vartheta\tau+j\tau-\tau)$$

$$(\nu=1,\ \ldots,\ r) \tag{1.26}$$

Let us now set the problem of bringing the system at the instant

$$t_1=j_1\tau \tag{1.27}$$

which is assumed to be a multiple of τ, to some given point $z_{p_\mu} = r_{p_\mu}$ ($\mu = 1, \ldots, m$) of the m-dimensional phase subspace $(z_{p_1}, \ldots, z_{p_m})$.

Let the number of additional external forces at our disposal also be equal to m and let these forces be $q_{s_i}(t), \ldots, q_{s_m}(t)$.

To solve this problem it is necessary to select some additional external forces $q_{s_i}(t)$ ($i = 1, \ldots, m$) such that the conditions

$$z_{p_\mu}(t_1)=r_{p_\mu}\qquad (\mu=1,\ \ldots,\ m) \tag{1.28}$$

are satisfied.

By taking the functions $q_{s_i}(t)$ ($i = 1, \ldots, m$) as step functions, the values of which are unchanged on the interval $(0,\ t_1)$

$$q_{s_i}(t)\equiv q_{s_i}(0)\qquad (0\leqslant t<t_1)\qquad (i=1,\ \ldots,\ m) \tag{1.29}$$

it will be found that the phase coordinates $z_\nu(t)$ ($\nu = 1, \ldots, r$) on the interval $(0,\ t_1)$ and the unknown values $q_{s_i}(0)$ ($i = 1, \ldots, m$) will be determined by the following system of equations:

$$z_\nu(t) = g_\nu(t) + \sum_{i=1}^{m} F_{\nu s_i}(t)\, q_{s_i}(0) + \sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{\nu l}(t, j\tau) \times$$

$$\times \psi_l(z_1(t - \vartheta\tau + j\tau - \tau),\ldots,\; z_r(t - \vartheta\tau + j\tau - \tau),\quad t - \vartheta\tau + j\tau - \tau)$$

$$(0 < t < j_1\tau) \qquad (\nu = 1, \ldots, r) \tag{1.30}$$

$$r_{p_\mu} - g_{p_\mu}(j_1\tau) = \sum_{i=1}^{m} F_{p_\mu s_i}(j_1\tau)\, q_{s_i}(0) + \sum_{l=1}^{n}\sum_{j=1}^{j_1} W_{p_\mu l}(j_1\tau, j\tau) \times$$

$$\times \psi_l(z_1(j\tau - \tau), \ldots, z_r(j\tau - \tau), j\tau - \tau) \qquad (\mu = 1, \ldots, m)$$

Here $F_{\nu s_i}(t)$ represents the known functions

$$F_{\nu s_i}(t) = \sum_{j=1}^{\vartheta} W_{\nu s_i}(t, j\tau) \qquad (\nu = 1, \ldots, r;\; i = 1, \ldots, m) \tag{1.31}$$

Equations (1.30) can be transformed in the following manner. From the second group of Equations (1.30 it appears that

$$q_{s_i}(0) = \frac{1}{M(j_1\tau)} K_{s_i}(j_1\tau) - \frac{1}{M(j_1\tau)} \sum_{\xi=1}^{m} A_{p_\xi s_i}(j_1\tau) \times \tag{1.32}$$

$$\times \sum_{l=1}^{n}\sum_{j=1}^{j_1} W_{p_\xi l}(j_1\tau, j\tau)\, \psi_l(z_1(j\tau - \tau), \ldots, z_r(j\tau - \tau), j\tau - \tau) \quad (i = 1, \ldots, m)$$

where

$$M(j_1\tau) = \begin{vmatrix} F_{p_1 s_1}(j_1\tau) & F_{p_1 s_2}(j_1\tau) & \ldots & F_{p_1 s_m}(j_1\tau) \\ \cdot & \cdot & \cdot & \cdot \\ F_{p_m s_1}(j_1\tau) & F_{p_m s_2}(j_1\tau) & \ldots & F_{p_m s_m}(j_1\tau) \end{vmatrix} \tag{1.33}$$

$$K_{s_i}(j_1\tau) = \sum_{\xi=1}^{m} A_{p_\xi s_i}(j_1\tau)[(r_{p_\xi} - g_{p_\xi})j_1\tau] \qquad (i = 1, \ldots, m) \tag{1.34}$$

and $A_{p_\xi s_i}(j_1\tau)$ (ξ, $i = 1, \ldots, m$) is the minor of the elements $F_{p_\xi s_i}$ in the determinant (1.33). Introducing the notation

$$k_{s_i}(j_1\tau) = \frac{1}{M(j_1\tau)} K_{s_i}(j_1\tau) \qquad (i = 1, \ldots, m) \tag{1.35}$$

$$U_{s_i l}(j_1\tau, j\tau) = \frac{1}{M(j_1\tau)} \sum_{\xi=1}^{m} A_{p_\xi s_i}(j_1\tau)\, W_{p_\xi l}(j_1\tau, j\tau) \quad \begin{pmatrix} i = 1, \ldots, m \\ l = 1, \ldots, n \end{pmatrix} \tag{1.36}$$

the Expression (1.32) can be put in the form

$$q_{s_i}(0) = k_{s_i}(j_1\tau) - \tag{1.37}$$
$$-\sum_{l=1}^{n}\sum_{j=1}^{j_1} U_{s_i l}(j_1\tau, j\tau)\,\psi_l(z_1(j\tau-\tau), \ldots, z_r(j\tau-\tau), j\tau-\tau) \quad (i = 1, \ldots, m)$$

Substituting the Expressions (1.37) found for $q_{s_i}(0)$ into the first group of Equations (1.30), the following system of nonlinear relations, relative to the unknown functions $z_\zeta(t)$ ($\zeta = 1, \ldots, r$) is obtained:

$$z_\nu(t) = G_\nu(t) - \sum_{i=1}^{m}\sum_{l=1}^{n} F_{\nu s_i}(t)\sum_{j=1}^{j_1} U_{s_i l}(j_1\tau, j\tau) \times \qquad \begin{pmatrix}0<t<j_1\tau\\ \nu=1,\ldots,r\end{pmatrix}$$
$$\times\psi_l(z_1(j\tau-\tau), \ldots, z_r(j\tau-\tau), j\tau-\tau) + \sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{\nu l}(t, j\tau)\times \tag{1.38}$$
$$\times\psi_l(z_1(t-\vartheta\tau+j\tau-\tau), \ldots, z_r(t-\vartheta\tau+j\tau-\tau), t-\vartheta\tau+j\tau-\tau)$$

where

$$G_\nu(t) = g_\nu(t) + \sum_{i=1}^{m} F_{\nu s_i}(t)\,k_{s_i}(j_1\tau) \qquad (\nu = 1, \ldots, r) \tag{1.39}$$

We notice that, as in [2], the number of equations constituting the system (1.38) decreases if the nonlinear functions Ψ_l ($l = 1, \ldots, n$) do not depend on some phase coordinates z_ρ. If, for instance, in the nonlinear function Ψ_l ($l = 1, \ldots, n$) there is only one phase coordinate z_k

$$\psi_l = \psi_l(z_k(t), t) \qquad (l = 1, \ldots, n) \tag{1.40}$$

then, in agreement with (1.38), it will be necessary to solve the following nonlinear equation with respect to the unknown function $z_k(t)$:

$$z_k(t) = G_k(t) - \sum_{i=1}^{m}\sum_{l=1}^{n} F_{\nu s_i}(t)\sum_{j=1}^{j_1} U_{s_i l}(j_1\tau, j\tau)\,\psi_l(z_k(j\tau-\tau), j\tau-\tau) + \tag{1.41}$$
$$+\sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{\nu l}(t, j\tau)\,\psi_l(z_k(t-\vartheta\tau+j\tau-\tau), t-\vartheta\tau+j\tau-\tau) \quad (0<t<j_1\tau)$$

The remaining phase coordinates z_ρ ($\rho = 1, \ldots, k-1, k+1, \ldots, r$) will be expressed as finite sums

$$z_\rho(t) = G_\rho(t) - \sum_{i=1}^{m}\sum_{l=1}^{n} F_{\rho s_i}(t)\sum_{j=1}^{j_1} U_{s_i l}(j_1\tau, j\tau)\,\psi_l(z_k(j\tau-\tau), j\tau-\tau) + \tag{1.42}$$

$$+\sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{\rho l}(t, j\tau)\,\psi_l(z_k(t-\vartheta\tau+j\tau-\tau), t-\vartheta\tau+j\tau-\tau) \quad (0<t<j_1\tau)$$

The additional external forces $q_{s_i}(t)$ $(i = 1, \ldots, m)$ will then, in agreement with (1.37), have the following form:

$$q_{s_i}(t) \equiv q_{s_i}(0) = k_{s_i}(j_1\tau) - \tag{1.43}$$

$$-\sum_{l=1}^{n}\sum_{j=1}^{j_1} U_{s_i l}(j_1\tau, j\tau)\,\psi_l(z_k(j\tau-\tau), j\tau-\tau) \quad (0<t<j_1\tau) \qquad (i=1, \ldots, m)$$

2. Let us now consider the case when the number of additional external forces $q_{s_i}(t)$ which can be realized in the controlled system is smaller than the order of the phase subspace to a given point of which the system has to be brought.

Let us assume that there is only one additional external force $q_s(t)$ available, the law of variation with time of which has to be chosen such that conditions (1.28) are satisfied, where in agreement with (1.27) $t_1 = j_1\tau$, and j_1 is some integer.

In order to solve the problem, the interval of time $(0, j_1\tau)$ will be divided into m equal or unequal subintervals $(0, \gamma_1\tau)$, $(\gamma_1\tau, \gamma_2\tau)$, $\ldots$, $(\gamma_{m-1}\tau, j_1\tau)$, where $\gamma_1, \ldots, \gamma_{m-1}$ are some integers. Let us take $q_s(t)$ as a step function and represent its values on each of these subintervals by $q_s(0)$, $q_s(\gamma_1\tau)$, $\ldots$, $q_s(\gamma_{m-1}\tau)$, respectively. Equations (1.26), determining the law of variation of the system, now take the form:

$$z_\nu(t) = g_\nu(t) + \sum_{i=0}^{m-1} q_s(\gamma_i\tau)\, 1(t-\gamma_i\tau) \sum_{j=\gamma_i+1}^{\sigma_i} W_{\nu s}(t, j\tau) + \sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{\nu l}(t, j\tau) \times$$

$$\times\, \psi_l(z_1(t-\vartheta\tau+j\tau-\tau), \ldots, z_r(t-\vartheta\tau+j\tau-\tau), t-\vartheta\tau+j\tau-\tau)$$

$$(0<t<j_1\tau) \qquad (\nu=1, \ldots, r) \tag{2.1}$$

where

$$\gamma_0 = 0, \qquad \gamma_m = j_1 \tag{2.2}$$

$$\sigma_i = \vartheta + (\gamma_{i+1} - \vartheta)\, 1(\vartheta - \gamma_{i+1}) \qquad (i = 0, 1, \ldots, m-1) \tag{2.3}$$

$$1(\zeta) = \begin{cases} 0 & \text{for } \zeta < 0 \\ 1 & \text{for } \zeta \geqslant 0 \end{cases} \tag{2.4}$$

Conditions (1.28) take on the form

$$r_{p_\mu} - g_{p_\mu}(j_1\tau) = \sum_{i=0}^{m-1} V_{p_\mu s}(\gamma_i\tau)\, q_s(\gamma_i\tau) + \\ + \sum_{l=1}^{n}\sum_{j=1}^{j_1} W_{p_\mu l}(j_1\tau, j\tau)\, \psi_l(z_1(j\tau - \tau), \ldots, z_r(j\tau - \tau), j\tau - \tau) \quad (\mu = 1, \ldots, m) \tag{2.5}$$

where

$$V_{p_\mu s}(\gamma_i\tau) = \sum_{j=\gamma_i+1}^{\gamma_{i+1}} W_{p_\mu s}(j_1\tau, j\tau) \qquad (\mu = 1, \ldots, m;\ i = 0, 1, \ldots, m-1) \tag{2.6}$$

From Equations (2.5) it follows that

$$q_s(\gamma_i\tau) = \varkappa_i(j_1\tau) - \qquad (i = 0, 1, \ldots, m-1) \\ - \sum_{l=1}^{n}\sum_{j=1}^{j_1} \Xi_{il}(j_1\tau, j\tau)\, \psi_l(z_1(j\tau - \tau), \ldots, z_r(j\tau - \tau), j\tau - \tau) \tag{2.7}$$

where

$$\varkappa_i(j_1\tau) = \frac{1}{\Lambda}\sum_{\xi=1}^{m} C_{p_\xi i}\,[r_{p_\xi} - g_{p_\xi}(j_1\tau)] \qquad (i = 0, 1, \ldots, m-1) \tag{2.8}$$

$$\Xi_{il}(j_1\tau, j\tau) = \frac{1}{\Lambda}\sum_{\xi=1}^{m} C_{p_\xi i} W_{p_\xi l}(j_1\tau, j\tau) \qquad (i = 0, 1, \ldots, m-1;\ l = 1, \ldots, n) \tag{2.9}$$

$$\Lambda = \begin{vmatrix} V_{p_1 s}(0) & V_{p_1 s}(\gamma_1\tau) & \ldots & V_{p_1 s}(\gamma_{m-1}\tau) \\ \cdots & \cdots & \cdots & \cdots \\ V_{p_m s}(0) & V_{p_m s}(\gamma_1\tau) & \ldots & V_{p_m s}(\gamma_{m-1}\tau) \end{vmatrix} \tag{2.10}$$

and $C_{p_\xi i}(\xi = 1, \ldots, m;\ i = 0, 1, \ldots, m-1)$ represent the minors of the elements $V_{p_\xi s}(\gamma_i\tau)$ in the determinant (2.10).

Substituting the Expressions (2.7) found for $q_s(\gamma_i\tau)$ in Equation (2.1), the following system of nonlinear equations with respect to the unknown functions $z_\zeta(\tau)$ is obtained:

$$z_\nu(t) = \Gamma_\nu(t) - \sum_{i=0}^{m-1}\sum_{l=1}^{n}\chi_{\nu i}(t)\sum_{j=1}^{j_1}\Xi_{il}(j_1\tau, j\tau)\times \qquad \begin{pmatrix}(0 \le t < j_1\tau)\\(\nu = 1, \ldots, r)\end{pmatrix} \tag{2.11}$$

$$\times\psi_l(z_1(j\tau - \tau), \ldots, z_r(j\tau - \tau), j\tau - \tau) + \sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{\nu l}(t, j\tau)\times$$

$$\times\psi_l(z_1(t - \vartheta\tau + j\tau - \tau), \ldots, z_r(t - \vartheta\tau + j\tau - \tau), t - \vartheta\tau + j\tau - \tau)$$

where

$$\Gamma_\nu(t) = g_\nu(t) + \sum_{i=0}^{m-1}\varkappa_i(j_1\tau)\chi_{\nu i}(t) \qquad (j = 1, \ldots, r) \tag{2.12}$$

$$\chi_{\nu i}(t) = 1(t - \gamma_i\tau)\sum_{j=\gamma_i+1}^{\sigma_i} W_{\nu s_i}(t, j\tau) \quad (j = 1, \ldots, r;\ i = 0, 1, \ldots, m-1) \tag{2.13}$$

In the case where the nonlinear functions ψ_l $(l = 1, \ldots, n)$ do not depend upon some of the phase coordinates z_ρ, the number of the nonlinear equations constituting the system (2.11) decreases. If, for instance, the functions ψ_l have the form (1.40) then, in agreement with (2.11), the following nonlinear relations with respect to the unknown function $z_k(t)$ are obtained:

$$z_k(t) = \Gamma_k(t) - \sum_{i=0}^{m-1}\sum_{l=1}^{n}\chi_{ki}(t)\sum_{j=1}^{j_1}\Xi_{il}(j_1\tau, j\tau)\psi_l(z_k(j\tau - \tau), j\tau - \tau) + \tag{2.14}$$

$$+\sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{kl}(t, j\tau)\psi_l(z_k(t - \vartheta\tau + j\tau - \tau), t - \vartheta\tau + j\tau - \tau) \quad (0 < t < j_1\tau)$$

The other phase coordinates will be expressed by the finite sums

$$z_\rho(t) = \Gamma_\rho(t) - \sum_{i=0}^{m-1}\sum_{l=1}^{n}\chi_{\rho i}(t)\sum_{j=1}^{j_1}\Xi_{il}(j_1\tau, j\tau)\psi_l(z_k(j\tau - \tau), j\tau - \tau) +$$

$$+\sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{\rho l}(t, j\tau)\psi_l(z_k(t - \vartheta\tau + j\tau - \tau), t - \vartheta\tau + j\tau - \tau) \tag{2.15}$$

$$(0 < t < j_1\tau) \qquad (\rho = 1, \ldots, k-1, k+1, \ldots, r)$$

The values of the additional external force $q_s(t)$ which appear as a step function in the intervals of time $(\gamma_i\tau, \gamma_{i+1}\tau)$ $(i = 0, 1, \ldots, m - 1)$, in agreement with (2.7) will be

$$q_s(\gamma_i\tau) = \varkappa_i(j_1\tau) - \\ - \sum_{l=1}^{n}\sum_{j=1}^{j_1} \Xi_{il}(j_1\tau, j\tau)\,\psi_l(z_k(j\tau - \tau), j\tau - \tau) \qquad (i = 0, 1, \ldots, m-1) \quad (2.16)$$

The described method makes it possible to realize the given law of motion in the m-dimensional subspace $(z_{p_1}, \ldots, z_{p_m})$, whereupon, if the number of the additional external forces $q_{s_i}(t)$ is smaller than the dimensions of the subspace, then conditions of type (1.28) will be fulfilled at the discrete points t_1, t_2, ...

To solve Equations (1.38) or (2.11), on the basis of which, in agreement with (1.37) and (2.7), the additional external forces $q_{s_i}(t)$ are determined, it is necessary to use numerical methods [5,6].

3. In the very simple case, where only the value of one phase coordinate z_p is assigned, and only one nonlinear function

$$\psi_\lambda = \psi_\lambda(z_k(t), t) \quad (3.1)$$

appears in the equation of motion, the additional external force $q_s(t)$ must be chosen such that the condition

$$z_p(t_1) = r_p \quad (3.2)$$

is satisfied.

The Equations (1.30) take now the form

$$z_\nu(t) = g_\nu(t) + F_{\nu s}(t)\,q_s(0) + \\ + \sum_{j=1}^{\vartheta} W_{\nu\lambda}(t, j\tau)\,\psi_\lambda(z_k(t - \vartheta\tau + j\tau - \tau), \quad t - \vartheta\tau + j\tau - \tau) \\ (0 < t < j_1\tau) \qquad (\nu = 1, \ldots, r) \quad (3.3)$$

$$r_p - g_p(j_1\tau) = F_{ps}(j_1\tau)\,q_s(0) + \sum_{j=1}^{j_1} W_{p\lambda}(j_1\tau, j\tau)\,\psi_\lambda(z_k(j\tau - \tau), j\tau - \tau)$$

where in agreement with (1.31)

$$F_{\nu s}(t) = \sum_{j=1}^{\vartheta} W_{\nu s}(t, j\tau) \qquad (\nu = 1, \ldots, r) \quad (3.4)$$

From the last Equation (3.3) it follows that

$$q_s(0) = \frac{1}{F_{ps}(j_1\tau)}\left[r_p - g_p(j_1\tau) - \sum_{i=1}^{j_1} W_{p\lambda}(j_1\tau,\ j\tau)\,\psi_\lambda(z_k(j\tau-\tau),\ j\tau-\tau)\right] \qquad (3.5)$$

Under these conditions, the first group of Equations (3.3) takes on the form

$$z_\nu(t) = g_\nu(t) + \frac{F_{\nu s}(t)}{F_{ps}(j_1\tau)}\left[r_p - g_p(j_1\tau) - \sum_{j=1}^{j_1} W_{p\lambda}(j_1\tau,\ j\tau)\,\psi_\lambda(z_k(j\tau-\tau),\ j\tau-\tau)\right] +$$
$$+ \sum_{j=1}^{\vartheta} W_{\nu\lambda}(t,\ j\tau)\,\psi_\lambda(z_k(t-\vartheta\tau+j\tau-\tau),\quad t-\vartheta\tau+j\tau-\tau) \qquad (3.6)$$
$$(0 < t < j_1\tau) \qquad (\nu = 1, \ldots, r)$$

In accordance with (3.6) we shall have the following nonlinear relation with respect to the unknown function $z_k(t)$:

$$z_k(t) = g_k(t) + \frac{F_{ks}(t)}{F_{ps}(j_1\tau)}\left[r_p - g_p(j_1\tau) - \sum_{j=1}^{j_1} W_{p\lambda}(j_1\tau,\ j\tau)\,\psi_\lambda(z_k(j\tau-\tau),\ j\tau-\tau)\right] +$$
$$+ \sum_{j=1}^{\vartheta} W_{k\lambda}(t,\ j\tau)\,\psi_\lambda(z_k(t-\vartheta\tau+j\tau-\tau),\ t-\vartheta\tau+j\tau-\tau) \qquad (3.7)$$
$$(0 < t < j_1\tau)$$

The remaining phase coordinates z_ρ ($\rho = 1, \ldots, k-1, k+1, \ldots, r$) will be expressed by finite sums

$$z_\rho(t) = g_\rho(t) + \frac{F_{\rho s}(t)}{F_{ps}(j_1\tau)}\left[r_p - g_p(j_1\tau) - \sum_{j=1}^{j_1} W_{p\lambda}(j_1\tau,\ j\tau)\,\psi_\lambda(z_k(j\tau-\tau),\ j\tau-\tau)\right] +$$
$$+ \sum_{j=1}^{\vartheta} W_{\rho\lambda}(t,\ j\tau)\,\psi_\lambda(z_k(t-\vartheta\tau+j\tau-\tau),\quad t-\vartheta\tau+j\tau-\tau) \qquad (3.8)$$
$$(0 < t < j_1\tau)$$

Here $q_s(t)$ is yet to be determined. In agreement with (1.29) and (3.5)

$$q_s(t) \equiv q\ \ (0) = k_s(j_1\tau) - \frac{1}{F_{ps}(j_1\tau)}\sum_{j=1}^{j_1} W_{p\lambda}(j_1\tau,\ j\tau)\,\psi_\lambda(z_k(j\tau-\tau),\ j\tau-\tau) \qquad (3.9)$$
$$(0 \leqslant t < t_1)$$

where

$$k_s(j_1\tau) = \frac{1}{F_{ps}(j_1\tau)}\left[r_p - g_p(j_1\tau)\right] \qquad (3.10)$$

The Expression (3.9) determines the additional external force $q_s(t)$ for the given particular case.

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